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A sufficient condition for Kim's conjecture on the competition numbers of graphs

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ABSTRACT

A *hole* of a graph G is an induced cycle of length at least 4. Kim (2005) [3] conjectured that the competition number $k(G)$ is bounded by $h(G) + 1$ for any graph G, where $h(G)$ is the number of holes of G. Li and Chang (2009) [5] proved that the conjecture is true for a graph whose holes all satisfy a property called 'independence'. In this paper, by using similar proof techniques in Li and Chang (2009) [5], we prove the conjecture for graphs satisfying two conditions that allow the holes to overlap a lot.

1. Introduction and preliminaries

Throughout the present paper, all graphs and digraphs (directed graphs) are finite and simple. We use $V(G)$ and $V(D)$ for the vertex set of a graph G or a digraph D. We use $E(G)$ for the edge set of a graph G. An edge with endpoints u and v is denoted by uv. We use $A(D)$ for the set of directed edges of a digraph D. Each element (u, v) of $A(D)$ is called an *arc* from u to v. A digraph is *acyclic* if it contains no directed cycles. The *competition graph* of a digraph D (see [7] for its background) is the graph $C(D)$ on $V(D)$ defined by

$$E(C(D)) = \{uv \mid \text{there is a vertex } x \in V(D) \text{ such that } (u, x), (v, x) \in A(D)\}.$$

For graphs G_1 and G_2, $G_1 \cup G_2$ is the graph defined by $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Let G be a graph and I_k a set of k isolated vertices, each of which is not a vertex of G, where k is a positive integer. Define I_0 to be the null graph, with no vertices. It is not difficult to see that there is an acyclic digraph D on $V(G) \cup I_{|E(G)|}$ such that $C(D) = G \cup I_{|E(G)|}$. The *competition number* of G, denoted $k(G)$, is defined by

$$k(G) = \min\{k \mid \text{there is an acyclic digraph } D \text{ such that } C(D) = G \cup I_k\}.$$

A *hole* of a graph G is an induced cycle of length at least 4. Researchers have observed a relation between the competition number $k(G)$ of G and the number $h(G)$ of holes of G. Roberts [7] proved that $k(G) \leq 1$ holds for a chordal graph G, that is, for a graph without holes (see Theorem 2.7). Cho and Kim [1] proved that $k(G) \leq 2$ holds for a graph G with exactly one hole. Kim [3] proposed the following conjecture.

Conjecture 1.1 (*Kim [3], 2005*)**.** *The competition number of a graph G is at most $h(G) + 1$.*

Li and Chang [6] proved that $k(G) \leq 3$ holds for a graph with exactly two holes. In terms of arrangement of holes, they studied a relation between the competition number of a graph and the number of holes of the graph. A hole C of G is *independent* if C satisfies the following conditions with respect to any other hole C' of G:

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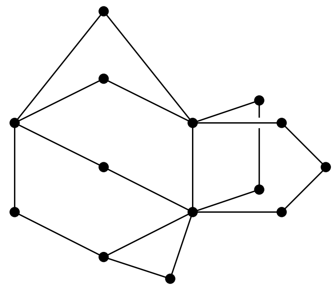

Fig. 1. A hole-simple graph that is not pairwise edge-disjoint and has a non-independent hole.

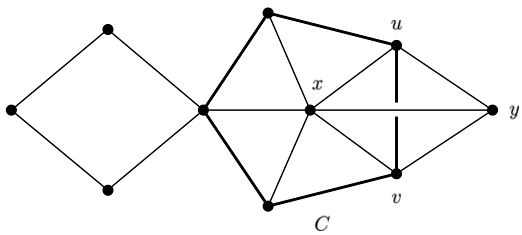

Fig. 2. A graph without Condition (2), whose holes all are independent and pairwise edge-disjoint.

(i) C and C' have at most two common vertices,
(ii) if C and C' have two common vertices, then they have one common edge, and
(iii) if there is a hole C' such that (ii) holds, then C has length at least 5.

Li and Chang [5] showed that $k(G) \leq h(G) + 1$ holds for a graph G whose holes all are independent. For a hole C of G, let X_C be the set of vertices adjacent to all vertices of C. It is not difficult to see that X_C is a clique for an independent hole C. Hence,

Condition (1) X_C is a clique of G for each hole C of G

holds when all holes are independent.

Other conditions on a graph G under which $k(G) \leq h(G) + 1$ holds were also given in [2,4]. The author [2] presented the following condition on graphs:

- for each hole C of a graph, there is an edge which is contained only in C among all induced cycles of the graph.

Kim et al. [4] considered graphs whose holes all are pairwise edge-disjoint. Such graphs are called *hole-edge-disjoint*. The above conditions on graphs imply that any distinct two holes overlap at most one edge.

We present a condition on graphs that allows the holes to overlap a lot. In this paper, a *walk W from u to v* means that u and v are not internal vertices of W. For a hole C of a graph G, a walk W from u to v is said to be *C-avoiding* if internal vertices of W are not in $V(C) \cup X_C$. For a hole C of G and $uv \in E(C)$, let

$$S_{C,uv} = \{x \in V(G) \mid x \text{ is an internal vertex of a } C\text{-avoiding walk from } u \text{ to } v\}, \quad \text{and}$$
$$T_{C,uv} = \{x \in V(G) \mid x \text{ is an internal vertex of a non-}C\text{-avoiding walk from } u \text{ to } v\}.$$

We consider the following condition for a graph G:

Condition (2) $S_{C,uv} \cap T_{C,uv} = \emptyset$ for any hole C of G and any edge $uv \in E(C)$.

A graph is *hole-simple* if it satisfies Conditions (1) and (2). By using similar proof techniques in [5], we prove that $k(G) \leq h(G)+1$ holds for hole-simple graphs. Hole-simple graphs may have holes that overlap a lot, violating the conditions required in [2,4,5].

The following two examples show that the condition in [4] that all holes are pairwise edge-disjoint, the condition in [5] that all holes are independent, and Conditions (1) and (2) do not imply each other.

Example 1.2. The graph in Fig. 1 is hole-simple, but it is not hole-edge-disjoint. It has a hole that is not independent. We can find two holes of length 6 that overlap at 4 edges.

Example 1.3. The graph in Fig. 2 is not hole-simple. It does not satisfy Condition (2), even though all holes are independent and pairwise edge-disjoint. Actually, for the edge uv of the hole C in the graph, the path (u, y, v) is C-avoiding, so the vertex y is in $S_{C,uv}$. Note that y is also in $T_{C,uv}$, because the walk (u, y, x, y, v) is non-C-avoiding.

In the rest of this section, we recall some basic notions on graphs. Let G be a graph. For a subset S of $V(G)$, the subgraph of G *induced by* S, denoted $G[S]$, is the subgraph with vertex set S whose edges consist of all edges of G joining two vertices of S. A subset X of $V(G)$ is a *clique* of G if X is a nonempty subset of $V(G)$ such that $G[X]$ is complete. For $u, v \in V(G)$ and $uv \notin E(G)$, let $G + uv$ be the graph with $V(G + uv) = V(G)$ and $E(G + uv) = E(G) \cup \{uv\}$. For $uv \in E(G)$, let $G - uv$ be the graph with $V(G - uv) = V(G)$ and $E(G - uv) = E(G) \setminus \{uv\}$. For $S \subseteq V(G)$, let $G - S = G[V(G) \setminus S]$. A subset $S \subset V(G)$ is called a *vertex cut* of G if $G - S$ has more components than G. Two paths from u to v are *disjoint* if u and v are their only common vertices.

2. The competition numbers of hole-simple graphs

For any clique Q of G, Li and Chang [5] investigated induced subgraphs G_1 and G_2 of G such that

- $G = G_1 \cup G_2$,
- $V(G_1) \cap V(G_2) = \{u, v\} \cup X_C$ is a clique in G, where C is a hole of G_1 and uv is an edge of C,
- Q is a clique in $G_1 - uv$, and
- both $G_1 - uv$ and G_2 have at most $h(G) - 1$ holes.

They constructed such induced subgraphs for graphs whose holes all are independent. We also construct such induced subgraphs for hole-simple graphs. Unless otherwise specified, in this section we consider a hole-simple graph G. For a hole C of G and $uv \in E(C)$, let $G_1 = G - S_{C,uv}$ and $G_2 = G[\{u, v\} \cup X_C \cup S_{C,uv}]$. Note that $V(G_1) \cap V(G_2) = \{u, v\} \cup X_C$. Under Condition (1), this set is a clique in G.

Lemma 2.1. *For a hole C in a graph G, no vertex of $S_{C,uv}$ is adjacent to a vertex of $V(G) \setminus (V(C) \cup X_C \cup S_{C,uv})$.*

Proof. If there are vertices $x \in S_{C,uv}$ and $y \in V(G) \setminus (V(C) \cup X_C \cup S_{C,uv})$ such that $xy \in E(G)$, then there is a C-avoiding walk W from u to v including x. Let W_1 from u to x and W_2 from x to v be subwalks of W. The walk W_1yW_2 is also a C-avoiding walk from u to v. Hence, $y \in S_{C,uv}$, a contradiction. □

Lemma 2.2. *For a hole C in a graph G satisfying Condition (2), no vertex of $S_{C,uv}$ is adjacent to a vertex of $V(C) \setminus \{u, v\}$.*

Proof. If there are vertices $x \in S_{C,uv}$ and $y \in V(C) \setminus \{u, v\}$ such that $xy \in E(G)$, then there is a C-avoiding walk W from u to v containing x. Let W_1 from u to x and W_2 from x to v be subwalks of W. The walk W_1yW_2 is non-C-avoiding, and hence $x \in T_{C,uv}$. This contradicts Condition (2). □

Lemmas 2.1 and 2.2 imply that $V(G_1) \cap V(G_2) = \{u, v\} \cup X_C$ is a vertex cut of G if $S_{C,uv} \neq \emptyset$. Hence, $G = G_1 \cup G_2$.

Remark 2.3. By the same argument as in the proof of Lemma 2.2, Condition (2) implies that no vertex of $S_{C,uv}$ is adjacent to a vertex of X_C, so no vertex of $S_{C,uv}$ is adjacent to a vertex of $X_C \cup V(C) \setminus \{u, v\}$. Thus, no vertex of $S_{C,uv}$ are adjacent to a vertex of $V(G) \setminus (\{u, v\} \cup S_{C,uv})$ under Condition (2). Note that $\{u, v\}$ is also a vertex cut of G if $S_{C,uv} \neq \emptyset$.

Lemma 2.4. *Let G be a graph with Condition (2) and C a hole of G. Let e and e' be edges of $E(C)$ such that $e \cap e' = \emptyset$. For any $x \in e \cup S_{C,e}$ and any $y \in e' \cup S_{C,e'}$, x is not adjacent to y in G unless $xy \in E(C)$. Also we have $(e \cup S_{C,e}) \cap (e' \cup S_{C,e'}) = \emptyset$.*

Proof. Note that $e' \cup S_{C,e'} \subset T_{C,e} \subset V(G) \setminus (e \cup S_{C,e})$ by Condition (2). Hence, $x \in S_{C,e}$ is not adjacent to $y \in e' \cup S_{C,e'}$ by Remark 2.3. We also reach contradictions in other cases by the same argument as above. □

Lemma 2.5. *Let G be a graph satisfying Condition (2). For any hole C and any edge $uv \in E(C)$, no pair (P_1, P_2) of disjoint paths from u to v satisfies the following conditions:*

- *P_i is a non-C-avoiding path for $i \in \{1, 2\}$ and*
- *$P_i + uv$ is a hole of G for $i \in \{1, 2\}$.*

Proof. Otherwise, for a hole C and $uv \in E(C)$, let P_1 and P_2 be two such paths from u to v. Note that P_i does not have vertices of X_C for $i \in \{1, 2\}$. If P_i has a vertex w_i of X_C, w_i is not in $\{u, v\}$ since $X_C \cap V(C) = \emptyset$. Now, either w_iu or w_iv is a chord of the hole $C_i = P_i + uv$ of G. Hence, P_i must have vertices of $V(C) \setminus \{u, v\}$ for $i \in \{1, 2\}$. Let x_i be a vertex of $V(C) \setminus \{u, v\}$ such that $x_i \in V(P_i)$ for $i \in \{1, 2\}$ (see Fig. 3). Now, P_2 is a C_1-avoiding path. Otherwise, P_2 must have a vertex z of X_{C_1} since P_1 and P_2 are disjoint. Now, either zu or zv is a chord of C_2, a contradiction.

Since we can take the subpaths P from x_1 to x_2 and P^{-1} from x_2 to x_1 of C without uv, the walk $P_2^uP^{-1}PP_2^v$ is a non-C_1-avoiding walk having vertices of $S_{C_1,uv}$, where P_2^u from u to x_2 and P_2^v from x_2 to v are the subpaths of P_2. This is contrary to Condition (2). □

Lemma 2.6. *Let G be a hole-simple graph and C a hole of G and $uv \in E(C)$. If $S_{C,uv} = \emptyset$, the graph $G - uv$ is also a hole-simple graph that satisfies $h(G - uv) < h(G)$.*

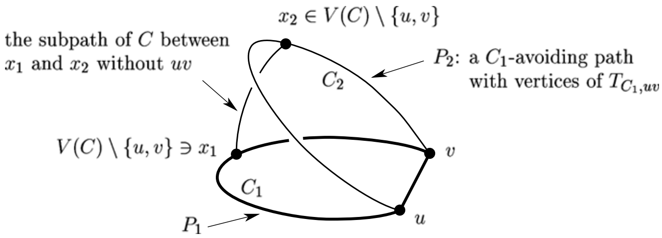

Fig. 3. Disjoint paths P_1 and P_2 from u to v, each of which is non-C-avoiding.

Proof. By the assumption $S_{C,uv} = \emptyset$, every walk from u to v is non-C-avoiding except for uv. We consider the family $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ of all non-C-avoiding paths from u to v, where

$$\mathcal{P}_1 = \{P \mid P \text{ is a non-}C\text{-avoiding path from } u \text{ to } v \text{ such that } P + uv \text{ is a hole of } G\}, \quad \text{and}$$
$$\mathcal{P}_2 = \{P \mid P \text{ is a non-}C\text{-avoiding path from } u \text{ to } v \text{ such that } P + uv \text{ is not a hole of } G\}.$$

The path P satisfying $P + uv = C$ is in $\mathcal{P}_1$. Note that a path in $\mathcal{P}_2$ does not form a hole of $G - uv$ with any other paths from u to v. Actually, for a non-C-avoiding path (u, x, v), the vertex x must be in X_C that is a clique in G by Condition (1). Hence, only pairs of disjoint paths in $\mathcal{P}_1$ may form holes of $G - uv$ that are not holes of G. By Lemma 2.5, all holes of $G - uv$ are holes of G. Thus, we have

$$h(G - uv) = h(G) - |\mathcal{P}_1| \leq h(G) - 1 < h(G).$$

The graph $G - uv$ satisfies Condition (2) since all holes of $G - uv$ are holes of G. We prove that $G - uv$ also satisfies Condition (1). Suppose to the contrary that there is a hole C' of $G - uv$ such that $\{u, v\} \subseteq X_{C'}$ holds. Note that $V(C) \cap X_C = \emptyset$ for each hole C.

If $V(C') \cap (V(C) \cup X_C) = \emptyset$, the path (u, x, v) is C-avoiding in G for any $x \in V(C')$, and hence $x \in S_{C,uv}$. This contradicts the assumption $S_{C,uv} = \emptyset$.

If $V(C') \cap (V(C) \cup X_C) \neq \emptyset$, let x be a vertex of $V(C') \cap (V(C) \cup X_C)$. Note that $V(C) \cap V(C') = \emptyset$. If not, we have a vertex $y \in V(C') \cap (V(C) \setminus \{u, v\})$. Then, either uy or vy is a chord of C in G, a contradiction. Thus, the vertex x is in $V(C') \cap X_C$. Note that the number of vertices of $V(C') \cap X_C$ is at most 2. Since C' is a cycle of length at least 4, there is a vertex $z \in V(C') \setminus X_C$. Hence, $z \in V(C')$ and $z \notin V(C) \cup X_C$ since $V(C) \cap V(C') = \emptyset$. Now, the path (u, z, v) is a C-avoiding in G, and hence $z \in S_{C,uv}$. This also contradicts the assumption $S_{C,uv} = \emptyset$.

Thus, $X_{C'}$ does not include $\{u, v\}$ for each hole C' of $G - uv$. This implies that $G - uv$ satisfies Condition (1). □

We need the following useful result to show our main theorem.

Theorem 2.7 (*Roberts [7]*). *Let G be a chordal graph and Q a clique of G. There is an acyclic digraph D such that $C(D) = G \cup I_1$ and all vertices of Q have only outgoing arcs of D.*

Theorem 2.8. *Let G be a hole-simple graph and Q a clique of G. There is an acyclic digraph D such that $C(D) = G \cup I_{h(G)+1}$ and all vertices of Q have only outgoing arcs in D. Consequently, $k(G) \leq h(G) + 1$.*

Proof. We prove this theorem by induction on the number of holes of a graph. If G is a graph without holes, that is, G is a chordal graph, the theorem is true for G by Theorem 2.7.

Let G be a hole-simple graph with at least one hole. Assume that the theorem is true for a hole-simple graph with at most $h(G) - 1$ holes.

Let C be a hole of G and e an edge of C. If $(e \cup S_{C,e}) \cap Q \neq \emptyset$, we may assume that $(e' \cup S_{C,e'}) \cap Q = \emptyset$ for $e' \in E(C)$ such that $e \cap e' = \emptyset$ by Lemma 2.4. Since C is a cycle of length at least 4, there is an edge uv of C such that $(\{u, v\} \cup S_{C,uv}) \cap Q = \emptyset$. Let $G_1 = G - S_{C,uv}$ and $G_2 = G[\{u, v\} \cup X_C \cup S_{C,uv}]$. By Lemmas 2.1 and 2.2 and Condition (1), the set $V(G_1) \cap V(G_2) = \{u, v\} \cup X_C$ is a clique vertex cut of G if $S_{C,uv} \neq \emptyset$. Hence, $G = G_1 \cup G_2$ and $h(G_2) = h(G) - h(G_1)$ since any hole of G belongs to either G_1 or G_2. Also note that both G_1 and G_2 are hole-simple.

Note that $h(G_2) < h(G)$ since G_2 does not include the hole C. Then, by the induction hypothesis, there is an acyclic digraph D_2 such that $C(D_2) = G_2 \cup I_{h(G_2)+1}$ and all vertices of the clique $\{u, v\} \cup X_C$ have only outgoing arcs in D_2. Note that C is a hole of G_1, Q is a clique of G_1, and $S_{C,uv} = \emptyset$ in G_1. By Lemma 2.6, the graph $G_1 - uv$ is hole-simple, and $h(G_1 - uv) < h(G_1) \leq h(G)$. By the induction hypothesis, there is an acyclic digraph D_1 such that $C(D_1) = (G_1 - uv) \cup I_{h(G_1 - uv)+1}$ and all vertices of the clique Q have only outgoing arcs in D_1. Now, consider the digraph D' with $V(D') = V(D_1) \cup V(D_2)$ and $A(D') = A(D_1) \cup A(D_2)$. Note that D' is an acyclic digraph such that $C(D') = G \cup I_{h(G_1 - uv)+h(G_2)+2}$ and all vertices of Q have only outgoing arcs in D', where $h(G_1 - uv) + h(G_2) \leq h(G_1) - 1 + h(G_2) = h(G) - 1$. Thus, $D = D' \cup I_{h(G_1) - 1 - h(G_1 - uv)}$ is a desired acyclic digraph. □

Remark 2.9. In the proof of Theorem 2.8, we may let $G_2 = G[\{u, v\} \cup S_{C,uv}]$ instead of letting $G_2 = G[\{u, v\} \cup X_C \cup S_{C,uv}]$ since $\{u, v\}$ is also a vertex cut of G under Condition (2) if $S_{C,uv} \neq \emptyset$ (see Remark 2.3).

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